

REGULAR SEQUENCES FROM DETERMINANTAL CONDITIONS

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ABSTRACT. In this paper we construct some regular sequences which arise naturally from determinantal conditions.

1. INTRODUCTION

Our study originated from the 1974 paper of J. Herzog [1] on the following theme: Let R be a Noetherian commutative ring with identity. Let $\underline{x} = \{x_1, \dots, x_n\}$ be a sequence in R . Let $\mathfrak{a} = (\alpha_{ij})$ be an $m \times n$ matrix with entries in R ; with $m \leq n$. A complex $D_*(\underline{x}, \mathfrak{a})$ was constructed in [1] with the following properties: If $n \geq 2$ and $m = n$; then $H_0(\underline{x}, \mathfrak{a})$ is isomorphic to $R/\langle a_1, \dots, a_n, \Delta \rangle$, where $a_i = \sum_{j=1}^n \alpha_{ij}x_j$, $i = 1, \dots, n$ and $\Delta = \det(\alpha_{ij})$. If $n \geq 3$ and $m = n - 1$, then $H_0(\underline{x}, \mathfrak{a})$ is isomorphic to $R/\langle a_1, \dots, a_n, \Delta^1, \dots, \Delta^n \rangle$, where $a_i = \sum_{j=1}^n \alpha_{ij}x_j$, $i = 1, \dots, n - 1$ and Δ^j is the determinant of the matrix obtained from \mathfrak{a} by deleting the j -th column. Acyclicity conditions on the complex $D_*(\underline{x}, \mathfrak{a})$ were derived in both the cases in order to decide perfectness and the Gorenstein property for the ideals $\langle a_1, \dots, a_n, \Delta \rangle$ and $\langle a_1, \dots, a_n, \Delta^1, \dots, \Delta^n \rangle$. Our aim is to study a generalised class of ideals of the form $I_1(XY)$ defined below.

Let $R = K[x_{ij}, y_{ij} \mid 1 \leq i, j \leq n]$, $n \geq 2$ and x_{ij} and y_{ij} are distinct indeterminates over the field K . Let $X = (x_{ij})_{n \times n}$ and $Y = (y_{ij})_{n \times n}$ be generic matrices. Let us write the product of the matrices X and Y as $XY = (f_{ij})_{n \times n}$, so that $f_{ij} = \sum_{k=1}^n x_{ik}y_{kj}$. Let $I_1(XY)$ denote the ideal generated by the polynomials f_{ij} , which are the 1×1 minors of the matrix XY . Certain properties like primality, primary decomposition and minimal free resolutions have been studied in [2], [3], [4].

A sequence of (homogeneous) polynomials p_1, p_2, \dots, p_r is called a *regular sequence* in R if the ideal $\langle p_1, p_2, \dots, p_r \rangle \neq R$ and if each p_i is non-zero

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divisor in $R/\langle p_1, p_2, \dots, p_{i-1} \rangle$, for every $1 \leq i \leq r$. It is easy to see that all the f_{ij} 's defined above do not form a regular sequence. For example, if $n = 2$ then $x_{12}y_{21}f_{11} + x_{11}y_{12}f_{12} - x_{22}y_{21}f_{21} = x_{21}y_{12}f_{22}$ shows that $f_{11}, f_{12}, f_{21}, f_{22}$ is not a regular sequence. We prove Theorem 3.1 to show that a subset can be chosen among these f_{ij} 's which form a regular sequence and they follow an interesting pattern. However, this is not a maximal one. The question which remains open is to find a subset of f_{ij} 's which form a regular sequence of maximal length.

2. TECHNICAL LEMMAS

Lemma 2.1. *Let $h_1, h_2, \dots, h_n \in R$ be such that with respect to a suitable monomial order on R the leading terms of them are mutually coprime. Then, h_1, h_2, \dots, h_n is a regular sequence in R .*

Proof. The element h_1 is a regular element in R , since R is a domain and $h_1 \neq 0$. By induction we assume that for $k \leq n-1$, $\{h_1, h_2, \dots, h_k\}$ forms a regular sequence in R . We note that the set $\{h_1, h_2, \dots, h_k\}$ is a Gröbner basis for the ideal J , since $\gcd(\text{Lt}(h_i), \text{Lt}(h_j)) = 1$ for every $i \neq j$. Let $gh_{k+1} \in J = \langle h_1, h_2, \dots, h_k \rangle$. Then $\text{Lt}(g)\text{Lt}(h_{k+1})$ must be divisible by $\text{Lt}(h_i)$ for some $1 \leq i \leq k$. But, $\gcd(\text{Lt}(h_i), \text{Lt}(h_{k+1})) = 1$, and hence $\text{Lt}(h_i)$ divides $\text{Lt}(g)$. Let $r = g - \frac{\text{Lt}(g)}{\text{Lt}(h_i)}h_i$. If $r = 0$, then $g \in J$. If $r \neq 0$, then $\text{Lt}(r) < \text{Lt}(g)$ and $rh_{k+1} \in J$. We follow the same argument with rh_{k+1} . \square

Lemma 2.2. *Let h_1, \dots, h_{n-1} be distinct polynomials and for $1 \leq r \leq n-1$ let m_1, \dots, m_{r+1} be distinct monomials in R . Suppose that the following properties are satisfied with respect to some monomial order on R :*

- (i) $\text{Lt}(h_i) = a_i b_i$ for every $1 \leq i \leq n-1$;
- (ii) $\gcd(\text{Lt}(h_i), \text{Lt}(h_j)) = 1$ for every $1 \leq i \neq j \leq n-1$;
- (iii) $\gcd(m_i, m_j) = 1$ for every $1 \leq i \neq j \leq r+1$;
- (iv) $\gcd(\text{Lt}(h_i), m_j) = 1$ for every $1 \leq i \leq n-1$ and $1 \leq j \leq r+1$.

Let $h_n = b_{k_1}m_1 + \dots + b_{k_r}m_r + m_{r+1} + (\text{lower order terms})$. Then

- (1) $h_1, \dots, h_{n-1}, m_1, \dots, m_r, h_n$ is a regular sequence.
- (2) Moreover, if g is a polynomial such that $\gcd(\text{Lt}(g), h_i) = 1$ for $1 \leq i \leq n-1$ and $\gcd(\text{Lt}(g), m_i) = 1$ for $1 \leq i \leq r+1$, then $h_1, \dots, h_{n-1}, m_1, \dots, m_r, h_n, g$ is a regular sequence.

Proof. (1) The sequence $h_1, \dots, h_{n-1}, m_1, \dots, m_r$ is a regular sequence by the Lemma 2.1. Let $\tilde{h}_n = h_n - \sum_{i=1}^r b_{k_i}m_i$, then $\text{Lt}(\tilde{h}_n) = m_{r+1}$ is coprime with $\text{Lt}(h_1), \dots, \text{Lt}(h_{n-1})$ and also coprime with m_1, \dots, m_r . Again

by the Lemma 2.1, we can write $h_1, \dots, h_{n-1}, m_1, \dots, m_r, \tilde{h}_n$ is a regular sequence.

Let $h_n \cdot p = \sum_{i=1}^{n-1} h_i p_i + \sum_{i=1}^r m_i q_i$, therefore

$$\tilde{h}_n \cdot p = \sum_{i=1}^{n-1} h_i p_i + \sum_{i=1}^r m_i (q_i - b_{k_i}).$$

This gives $p \in \langle h_1, \dots, h_{n-1}, m_1, \dots, m_r \rangle$, since $h_1, \dots, h_{n-1}, m_1, \dots, m_r, \tilde{h}_n$ is a regular sequence. Therefore, $h_1, \dots, h_{n-1}, m_1, \dots, m_r, h_n$ is a regular sequence.

(2) Let $g \cdot p' = \sum_{i=1}^n h_i p'_i + \sum_{i=1}^r m_i q'_i$, therefore

$$g \cdot p' = \sum_{i=1}^{n-1} h_i p'_i + \sum_{i=1}^r m_{k_i} (q'_i + b_{k_i}) + \tilde{h}_n p'_n.$$

Now $h_1, \dots, h_{n-1}, m_1, \dots, m_r, \tilde{h}_n, g$ is a regular sequence by 2.1; hence $p' \in \langle h_1, \dots, h_{n-1}, m_{k_1}, \dots, m_r, \tilde{h}_n \rangle$. \square

3. THE MAIN THEOREM

Theorem 3.1. Suppose that $k_t = 1 + \left(\left\lceil \frac{n}{t} \right\rceil - 1\right)t$, then $k_n = 1$. The ordered set

$$\mathcal{F} = \{f_{11}, \dots, f_{n1}\} \cup \{f_{12}, f_{32}, \dots, f_{k_2 2}\} \cup \{f_{13}, f_{43}, \dots, f_{k_3 3}\} \cup \dots \cup \{f_{1n}\}$$

is a regular sequence in R . The polynomials occurring in the list follow the pattern indicated below:

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & \cdots & f_{1n} \\ f_{21} & \times & \times & \times & \cdots & \times \\ f_{31} & f_{32} & \times & \times & \cdots & \times \\ f_{41} & \times & f_{43} & \times & \cdots & \times \\ f_{51} & f_{52} & \times & f_{54} & \cdots & \times \\ f_{61} & \times & \times & \times & \cdots & \times \\ f_{71} & f_{72} & f_{73} & \times & \cdots & \times \\ f_{81} & \times & \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

Proof. Consider the lexicographic monomial order given by

$$\begin{aligned}
 x_{11} \cdots > x_{nn} &> x_{12} > x_{23} > \cdots > x_{(n-1)n} \\
 &> x_{13} > x_{24} > \cdots > x_{(n-2)n} \\
 &> \vdots \\
 &> x_{1n} \\
 &> x_{ij} > y_{kl}, \quad \text{for all } i > j \quad \text{and} \quad 1 \leq k, l \leq n.
 \end{aligned}$$

In order to show that the set \mathcal{F} is a regular sequence we consider a larger ordered set of polynomials by adding some indeterminates in the list so that they follow the properties listed in Lemma 2.2. The new ordered set of polynomials we consider is

$$\begin{aligned}
 \tilde{\mathcal{F}} = & \{f_{11}, \dots, f_{n1}\} \cup \\
 & \{y_{12}, f_{12}, y_{32}, f_{32}, \dots, y_{k_2 2}, f_{k_2 2}\} \cup \\
 & \{y_{13}, y_{23}, f_{13}, \dots, y_{k_3 3}, y_{(k_3+1)3}, f_{k_3 3}\} \cup \\
 & \vdots \\
 & \{y_{1t}, \dots, y_{(t-1)t}, f_{1t}, \dots, f_{k_t t}\} \cup \\
 & \vdots \\
 & \{y_{1n}, \dots, y_{(n-1)n}, f_{1n}\}.
 \end{aligned}$$

If we can show that $\tilde{\mathcal{F}}$ is a regular sequence in this order, then, under any permutation of this order the polynomials would still form a regular sequence because of homogeneity. Therefore, we can rearrange the entries of $\tilde{\mathcal{F}}$ in such a way that the elements that appear first are the ones listed in \mathcal{F} and that proves our claim. By Lemma 2.1, $f_{11}, \dots, f_{n1}, y_{12}$ is a regular sequence. Using Lemma 2.2 we can add $f_{12} = x_{11}y_{12} + x_{12}y_{22} + \sum_{k=3}^n x_{1k}y_{k2}$ in the list. Therefore, $f_{11}, \dots, f_{n1}, y_{12}, f_{12}$ is a regular sequence. The indeterminate y_{32} does not divide the any of the leading terms of $f_{11}, \dots, f_{n1}, y_{12}, f_{12} - x_{11}y_{12}$. Therefore, $f_{11}, \dots, f_{n1}, y_{12}, f_{12}, y_{32}$ is a regular sequence, by Lemma 2.2. Therefore, $\tilde{\mathcal{F}}$ is a regular sequence if we continue the same process. The above comment proves that \mathcal{F} is a regular sequence. \square

REFERENCES

- [1] J. Herzog, Certain Complexes Associated to a Sequence and a Matrix, *Manuscripta Math.* 12(1974) 217–248.
- [2] J. Saha, I. Sengupta, G. Tripathi, Ideals of the form $I_1(XY)$, *arXiv:1609.02765v2 [math.AC]* 2016.
- [3] J. Saha, I. Sengupta, G. Tripathi, Primality of Certain Determinantal Ideals, *arXiv:1610.00926v2 [math.AC]* 2016.

- [4] J. Saha, I. Sengupta, G. Tripathi, Betti numbers of certain Sum Ideals, *arXiv:1611.04732v1 [math.AC]* 2016.

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